DETERMINATION OF THE LOWEST POINT OF THE CONDUCTOR IN INCLINED SPANS BASED ON A KNOWN MAXIMAL SAG OF THE PARABOLA

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ABSTRACT

The lowest point of the conductor is one specific point, but it is often the most critical point too, so it deserves to be discussed separately. In levelled spans it is located at a mid-span, while in inclined spans it moves towards the lower suspension point. It means that the position of the lowest point of the conductor has to be determined in each inclined span separately. Besides, knowing the configuration of the terrain it is also an important prior condition of a precise planning overhead power lines.
This paper shows three entirely different methods for the determination of the lowest point of the conductor in inclined spans on the basis of a known maximal sag, which refers to the chosen conductor type, span length, tension and temperature. Presented methods can be used for spans approximately up to 400 metres, with the non-significant difference in elevation between the suspension points, since these calculations are parabola based.

INTRODUCTION

The sag–tension calculation [1] is well described in literature and it gives the value of the maximal sag of the parabola, necessary for drawing the conductor curve. Generally the coordinate system with the origin at the vertex (the lowest point) of the conductor curve is in use, according to the existing literature. However, when designing overhead lines (figure 1.) it is more advantageous to measure the distance from the left–hand side support toward the right–hand side one than to measure it from the lowest point toward the supports of the span. For that reason, the equation for the conductor curve will be derived in a new coordinate system with an origin that is put on the line of the left–hand side support, on the elevation of the bottom of the lower–standing support of the span. This way the y–coordinate of the conductor curve presents the conductor height related to x–axis, but its x–coordinate presents the horizontal distance from the left–hand side support. (When calculating the clearance the height of the terrain related to x–axis has to be taken into consideration.) The provided equation for the conductor curve this way gives different mathematical solutions for the determination of the lowest point of the conductor:

1. Derivative of the conductor curve,
2. Finding the biggest levelled span within the inclined one,
3. Transforming parabola equation into its vertex form.

A concrete application for designing overhead lines will be shown by numerical examples both for inclined spans ($h_1 \neq h_2$) and levelled spans ($h_1 = h_2$) as well.

Figure 1. *overhead power line*

The equation for the conductor curve and the lowest point of the conductor are derived with the help of figure 2. which contains all the sufficient points, lines and the curve in a common coordinate system for all shown methods. In aim to provide a universal algorithm, an inclined span has been used, while the levelled span is actually its simplified case. The necessary data for the calculations are: the span length, heights of the suspension points related to the x–axis and the maximal sag.

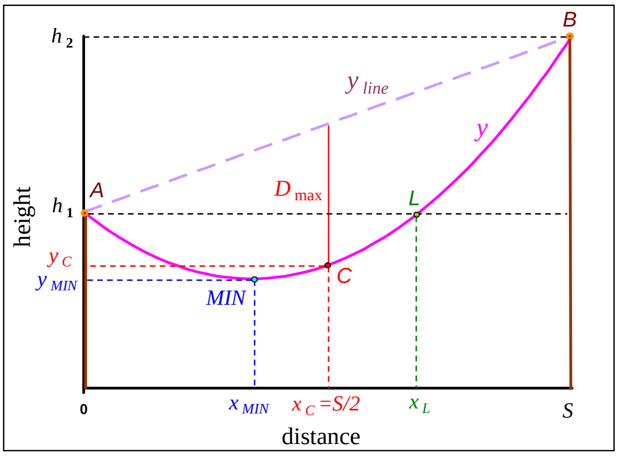

Figure 2. *inclined span*

S – span length,
h_1 – height of the left–hand side suspension point A,
h_2 – height of the right–hand side suspension point B,
D_{max} – maximal sag,
y_{line} – straight line between the suspension points A and B,
y – conductor curve (parabola),
$MIN(x_{MIN}; y_{MIN})$ – the lowest point of the conductor curve,
$C(S/2; y_c)$ – maximal sag point of the conductor,
L – conductor point for finding the biggest levelled span within an inclined one.

EQUATION FOR THE CONDUCTOR CURVE

The common condition to describe the each listed method for the determination of the lowest point of the conductor is a necessary previous definition of the equation for the conductor curve. This task is solved by a new method, using three points of the conductor, as the parabola is completely defined when any three points of its curve are known. Two suspension points of the span $A(0;h_1)$ and $B(S;h_2)$ are always known points, while the third necessary point (C) is defined by the known maximal sag. The feature of the parabola is being used that its maximal sag is always located in the middle of the span, i.e. both in case of levelled and inclined spans [2]. (For the catenary it is different.) So the x–coordinate of the point C (2) is known ($x_C=S/2$), but its y–coordinate can be obtained by (1).

$$y_C = \frac{h_1 + h_2}{2} - D_{\max} \tag{1}$$

$$C\left(\frac{S}{2}; \frac{h_1 + h_2}{2} - D_{\max}\right) \tag{2}$$

Based on three points A, B, C of the parabola curve, the system of three algebraic equations in three unknowns [3] can be written in a matrix form (4), by utilizing the parabola equation in a standard form (3).

$$y = ax^2 + bx + c \tag{3}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ S^2 & S & 1 \\ (S/2)^2 & S/2 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ (h_1 + h_2)/2 - D_{\max} \end{bmatrix} \tag{4}$$

The solution of this system is (5) and it presents coefficients a, b, c of the parabola equation (3).

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4D_{\max}/S^2 \\ (h_2 - h_1 - 4D_{\max})/S \\ h_1 \end{bmatrix} \tag{5}$$

After substituting a, b, c into (3), the equation for the conductor curve is derived:

$$y = \frac{4D_{\max}}{S^2} x^2 + \frac{h_2 - h_1 - 4D_{\max}}{S} x + h_1 \qquad x \in [0, S] \tag{6}$$

The equation (6) is universal, since it is usable in case of any type of inclined spans ($h_1<h_2$ or $h_1>h_2$) and in case of levelled spans ($h_1=h_2$) as well.

Having the minimum turning point, the conductor curve is a cup–shaped parabola. So, according to the parabola rules the coefficient a should be positive. From (6) it is obvious that it is always so, since the span length and the maximal sag are also always positive. The equation (6) is a standard form of the parabola equation for the conductor curve.

THE LOWEST POINT OF THE CONDUCTOR

Once the equation for the conductor curve is derived there are different mathematical solutions to define the lowest point of the conductor on the basis of a known maximal sag of the parabola. The validity of the following three methods is proved by their identical results.

Derivative of the conductor curve

The basic way to find the x–coordinate of the extreme point (minimum or maximum) of the curve $y(x)$ is to find the first derivative dy/dx and to solve the equation $dy/dx=0$. Then by substituting the result into the equation of the curve the y–coordinate of the extreme point is defined too. The application of this method on the conductor curve shown by (7)–(9) yields the expression (10) for the determination of the lowest point of the conductor.

$$\frac{dy}{dx} = \frac{4D_{\max}}{S^2} x + \frac{h_2 - h_1 - 4D_{\max}}{S} \tag{7}$$

$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad x_{MIN} = \frac{S}{2}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right) \tag{8}$$

$$y_{MIN} = y(x_{MIN}) = h_1 - D_{\max}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right)^2 \tag{9}$$

$$MIN\left(\frac{S}{2}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right); h_1 - D_{\max}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right)^2\right) \tag{10}$$

Finding the biggest levelled span within an inclined span

From figure 2. it can be seen that there is one point (denoted by L) on the conductor curve which lies on the same elevation (h_1) as the suspension point A does (11). By determination of the x–coordinate of the point L, the x–coordinate of the point MIN can be easily defined too, since $x_{MIN}=x_L/2$. The x_L is actually the biggest levelled span within the given inclined one. The algorithm for finding x_{MIN} (16) is shown in the following lines:

$$y_A = y_L = h_1 \tag{11}$$

$$h_1 = \frac{4D_{\max}}{S^2} x_L^{\ 2} + \frac{h_2 - h_1 - 4D_{\max}}{S} x_L + h_1 \tag{12}$$

$$x_L\left[4D_{\max} x_L + S(h_2 - h_1) - 4SD_{\max}\right] = 0 \tag{13}$$

The $x_L=0$ is not an appropriate solution, so we have to solve the equation (14) to get x_L (15):

$$4D_{\max} x_L + S(h_2 - h_1) - 4SD_{\max} = 0 \tag{14}$$

$$x_L = S\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right) \tag{15}$$

$$x_{MIN} = \frac{x_L}{2} = \frac{S}{2}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right) \tag{16}$$

Since the x–coordinate of the point MIN is obtained, its y–coordinate can be defined by the same way from the previous method, i.e. by (9). The latter two algorithms are provided for case $h_1<h_2$, but the case $h_1>h_2$ also produces the same results.

Transforming parabola equation into vertex form

The lowest point of the conductor is actually a local extreme, i.e. the vertex point of the parabola. Considering that each parabola equation in a standard form can be written in the vertex form (17) too, it can be practically used to find the coordinates of the vertex. With the help of the expressions (18)–(20) it is easy to transform the previously derived equation for the conductor curve from its standard form (6) into the vertex form (21), as in the following:

$$y = ax^2 + bx + c = a(x - x_{MIN})^2 + y_{MIN} \tag{17}$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \tag{18}$$

$$x_{MIN} = -\frac{b}{2a} = \frac{S}{2}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right) \tag{19}$$

$$y_{MIN} = \frac{4ac - b^2}{4a} = h_1 - D_{\max}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right)^2 \tag{20}$$

$$y = \frac{4D_{\max}}{S^2}\left[x - \frac{S}{2}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right)\right]^2 + h_1 - D_{\max}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right)^2 \tag{21}$$

The latter equation has an advantage in comparison to its standard form since it can be used for horizontal or/and vertical replacement of the curve.

Simple case: the lowest point in a mid–span

The simplification of an inclined span is a levelled span, with the suspension points of the conductor on the same elevation, i.e. h_1=h_2=h. In this case the conductor curve equation is simpler, since the lowest point of the conductor *MIN* (22) is located in the middle of the span. The standard form of the conductor curve equation is now (23), but its vertex form is (24).

$$MIN(S/2; h - D_{\max}) \tag{22}$$

$$y = \frac{4D_{\max}}{S^2}x^2 - \frac{4D_{\max}}{S}x + h \qquad x \in [0, S] \tag{23}$$

$$y = \frac{4D_{\max}}{S^2}\left(x - \frac{S}{2}\right)^2 + h - D_{\max} \qquad x \in [0, S] \tag{24}$$

Special case: the lowest point of the conductor differs from the vertex point of the parabola

Let us mention that the vertex of the parabola and the lowest point of the conductor are generally the same point (point *MIN*) and in that case $x_{MIN} \in [0, S]$, like in figure 2.

Figure 3. shows one special case of an inclined span when the vertex point (*MIN*) is out of the span, i.e. $x_{MIN} \notin [0, S]$.

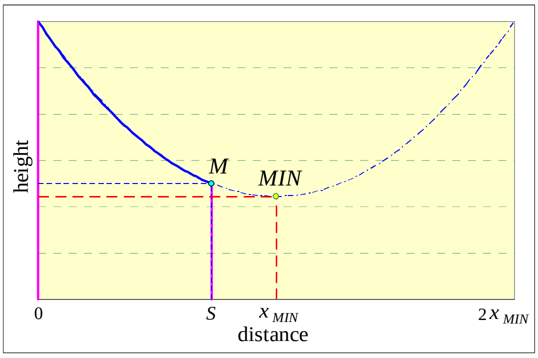

Figure 3. *inclined span with $x_{MIN} > S$*

In this case the lowest point of the conductor (now denoted by *M*) is identical with the lower suspension point of the span, but the vertex point is still given by (10). To present this rare case appropriately, the parabola curve is shown on the interval $[0, 2x_{MIN}]$ and drawn by a broken line, but the conductor curve is still defined only on the interval $[0,S]$. In this example $x_{MIN} > S$ and $h_1 > h_2$. There is another type of the special case too, when $x_{MIN} < 0$ and $h_1 < h_2$.

APPLICATION OF THE ALGORITHMS

The usefulness and applicability of the shown algorithms in practice will be presented by three similar examples with the same span length and a maximal sag, but with differences in heights of the suspension points in each example. The input data are given in table 1.

Table 1. Input data for examples 1, 2, 3

	Example 1 $h_1 < h_2$	Example 2 $h_1 = h_2$	Example 3 $h_1 > h_2$
S [m]	200	200	200
h_1 [m]	14	22	30
h_2 [m]	30	22	14
$D_{\max}$ [m]	8	8	8

Task: The coordinates of the lowest point of the conductor curve and the coefficient *a* of the parabola have to be calculated in each example. Use the results to write the equations for the conductor curves in the vertex form of the parabola equation. Finally, define the vertex form of the sag equation for computing the conductor sag at an arbitrary point of the span. Draw its curve together with the conductor curves on the same diagram in the common coordinate system.

Solution: The vertex point and coefficient *a* can be easily obtained by the application of (17)–(20). The following table contains the provided results for each example.

Table 2. Results from examples 1, 2, 3

	Example 1	Example 2	Example 3
a [m]	0,0008	0,0008	0,0008
x_{MIN} [m]	50	100	150
y_{MIN} [m]	12	14	12

Based on table 2. the equations for the conductor curves are the following:

$$y_1 = 8 \cdot 10^{-4}(x - 50)^2 + 12 \qquad x \in [0,200] \tag{25}$$

$$y_2 = 8 \cdot 10^{-4}(x - 100)^2 + 14 \qquad x \in [0,200] \tag{26}$$

$$y_3 = 8 \cdot 10^{-4}(x - 150)^2 + 12 \qquad x \in [0,200] \tag{27}$$

According to the sag definition, the equation for the sag is:

$$D(x) = y_{line} - y \qquad x \in [0, S] \tag{28}$$

So, first we have to write the equations for straight lines:

$$y_{line\ 1} = 0{,}08x + 14 \qquad x \in [0,200] \tag{29}$$

$$y_{line\ 2} = 22 \tag{30}$$

$$y_{line\ 3} = -0{,}08x + 30 \qquad x \in [0,200] \tag{31}$$

The application of (28) for the three given examples gives the same parabolic equation for the conductor sag (32).

$$D(x) = D_1(x) = D_2(x) = D_3(x) = -8 \cdot 10^{-4}(x-100)^2 + 8 \quad (32)$$

This means that with the equal span length and with the equal coefficient a both in levelled and inclined spans, the sag value in two cases is equal in each point of the span. This conclusion can be written in the following relation:

$$\text{if} \quad S_{inc} = S_{lev} \;\wedge\; a_{inc} = a_{lev} \;\Rightarrow\; D_{inc}(x) \equiv D_{lev}(x) \quad (33)$$

In equation (32) the coefficient a of the parabola is negative, hence the sag curve is a hat–shaped parabola. Now all the results from the given task can be appropriately presented on the common diagram (figure 4.).

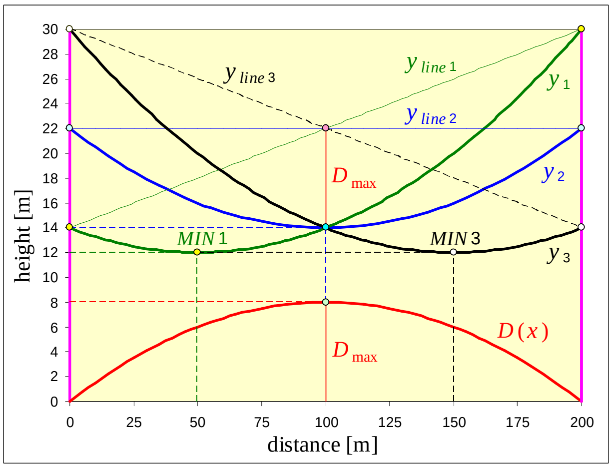

Figure 4. *sag and conductor curves from examples 1–3.*

CONDUCTOR SAG AT THE LOWEST POINT

By using (28), the expression for the sag calculation at an arbitrary point of the span can be defined (34).

$$D(x) = \frac{-4D_{max}}{S^2}\left(x - \frac{S}{2}\right)^2 + D_{max} \qquad x \in [0, S] \quad (34)$$

It can be applied to derive the formula for calculation the sag value at the lowest point of the conductor by substitution x_{MIN} into (34).

$$D(x_{MIN}) = D_{max}\left[1 - \left(\frac{h_2 - h_1}{4D_{max}}\right)^2\right] \quad (35)$$

The following relations concern to the low–point sag:

$$0 \le x_{MIN} < S/2 \;\Rightarrow\; 0 \le D(x_{MIN}) < D_{max} \quad (36)$$

$$x_{MIN} = S/2 \;\Rightarrow\; D(x_{MIN}) = D_{max} \quad (37)$$

$$S/2 < x_{MIN} \le S \;\Rightarrow\; D_{max} > D(x_{MIN}) \ge 0 \quad (38)$$

$$0 > x_{MIN} > S \;\Rightarrow\; D(x_{MIN}) \text{ is not defined.} \quad (39)$$

The latter relation corresponds to special cases of inclined spans when the vertex is out of the span, so it is not identical to the lowest point of the conductor. Since in this case the latter is identical to the lower suspension point of the span, the sag value at that point is equal to zero.

COMPARING THE CATENARY LOW POINT

The catenary equation for the conductor curve in the new coordinate system mentioned in this paper is given by (40) [4], where c is the catenary constant.

$$y_{cat} = 2c \cdot sh^2 \frac{x - x_{MIN}}{2c} + y_{MIN} \qquad x \in [0, S] \quad (40)$$

The coordinates of the vertex (i.e. the lowest point) of the catenary are (41) [1] and (42) [4].

$$x_{MIN\ cat} = \frac{S}{2} - c \cdot arsh\frac{h_2 - h_1}{2c \cdot sh(S/2c)} \quad (41)$$

$$y_{MIN\ cat} = h_1 - 2c \cdot sh^2\left\{\frac{1}{2} \cdot \left[\frac{S}{2c} - arsh\frac{h_2 - h_1}{2c \cdot sh(S/2c)}\right]\right\} \quad (42)$$

It is obvious that the computation of the lowest point of the catenary is more complicated in comparison to the parabola. However, it has to be mentioned that the application of the catenary based calculation has no limitations, but the parabola based calculation does.

CONCLUSION

In comparison to the existing literature, this paper uses a non-standard coordinate system for the conductor curve. However, it is more natural in practice, since when designing overhead lines the distance is always measured from the left–hand side support of the span, but not from the conductor low–point. The new approach gives more mathematical solutions for solving different tasks. As it has been shown, the coordinates of the lowest point of the conductor are determined algebraically by three methods. The provided expression for the x–coordinate is naturally identical to the appropriate one from literature [5], but now it is derived from the equation for the conductor curve in a new coordinate system. In addition, the y–coordinate of the lowest point related to x–axis and the formula for the sag at the lowest point of the conductor are given.

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